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h-fold sums from a set with few products

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Abstract: For any $h \in \mathbb{N}$ and for any set $A \subset \mathbb{C}$, $|A| = n$, satisfying

$$|AA| \ll n^{1+\varepsilon}$$

we have

$$|hA| \gg n^{\log(h)/(2\log 2)+O(1)}$$

provided that $\varepsilon = O(1/h)$.

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1. Statement of results

Given a finite subset A of some field, we let $A+A$, AA, and A/A denote the set of sums $a+b$, products ab, and ratios a/b, respectively, where $a, b \in A$. For $k \in \mathbb{N}$, we let kA denote the k-fold sumset $A+\ldots+A$.

Erdős and Szemerédi [7] proved that for some $\varepsilon > 0$ and $n \geqslant n_0(\varepsilon)$, we have that for any set A for n real numbers, either the sumset $A+A$ or the product set AA has at least $n^{1+\varepsilon}$ elements and conjectured that this is true for any $\varepsilon < 1$. The problem is still open. The best result was achieved by Solymosi [10] who proved the conjecture for $\varepsilon < 1/3$. Also, the interested reader can find in [10] the history of the study of the problem. The result of [10] was extended to sets of complex numbers in [8].

Bourgain and Chang [1] proved that if $A \subset \mathbb{Z}$, $k \in \mathbb{N}$,

$$|AA| \leqslant |A|^{1+\varepsilon}, \tag{1.1}$$

then $|kA| \gg_{k,\varepsilon} |A|^{k-\delta}$, where $\delta = \delta(\varepsilon, k) \to 0$ as k is fixed and $\varepsilon \to 0$. In [2] they extended this result to subsets of $\mathbb{C}$ consisting of algebraic integers of bounded degree.

For arbitrary finite $A \subset \mathbb{R}$ or $A \subset \mathbb{C}$ the conclusion of the same strength is known only under a stronger supposition on $|AA|$. Chipeniuk [5] for $A \subset \mathbb{C}$ proved that $|kA| \gg_k |A|^k$ provided that $|AA| \ll_k |A| \log |A|$. Earlier similar results were obtained by Chang for $A \subset \mathbb{Z}$ [3] and for $A \subset \mathbb{R}$ [4].

If for $A \subset \mathbb{R}$ we weaken the supposition on $|AA|$ to (1.1) for some small ε, then less is known about lower estimates for kA. Croot and Hart [6] proved the following.

Theorem A. *For all $h \geqslant 2$ and $0 < \varepsilon < \varepsilon_0(h)$ we have that the following holds for all $n > n_0(h, \varepsilon)$: if A is a set of n real numbers and*

$$|AA| \leqslant n^{1+\varepsilon},$$

then

$$|hA| \geqslant n^{\log(h/2)/2\log 2 + 1/2 - f_h(\varepsilon)},$$

where $f_h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The aim of this paper is to indicate for which $\varepsilon > 0$ one can take $f_h(\varepsilon) \ll 1$ and to extend Theorem A to subsets of complex numbers. The main result is the following.

Theorem 1. *For any $k \in \mathbb{Z}_+$, $K \geqslant 1$, and for any set $A \subset \mathbb{C}$, $|A| = n$, satisfying*

$$|AA| \leqslant Kn$$

we have

$$|2^k A| \geqslant c(k) n^{1+k/2} K^{-u_k},$$

where

$$u_k = \frac{1}{2}\left(2^{k+2} - k - 4\right)$$

and $c(k) > 0$.

In particular, we have

$$|2^k A| \gg n^{k/2+O(1)}$$

for $K = n^{\varepsilon}$, $\varepsilon \ll 2^{-k}$, and

$$|2^k A| \gg n^{1+k/2+O(1)}$$

for $K = n^{\varepsilon}$, $\varepsilon = O\left(2^{-k}\right)$.

To estimate $|hA|$ for an arbitrary $h \in \mathbb{N}$, we choose k so that

$$2^k \leqslant h < 2^{k+1}$$

and use the inequality $|hA| \geqslant |2^k A|$. We see that in Theorem A one can take $f_h(\varepsilon) = O(1)$ if $\varepsilon = O\left(h^{-1}\right)$.

The proof of Theorem 1 is based on the arguments from [10] (sufficient to prove it for $A \subset \mathbb{R}$) and [8].

2. The proof of Theorem 1

Without loss of generality we may assume that the set $A \subset \mathbb{C} \setminus \{0\}$ is located in a reasonably small angular sector $|\tan(2 \arg z)| < \eta$ around the real axis, with the vertex at 0. The constant $\eta > 0$ does not go to zero: it only needs to be small enough for the geometric argument in the proof of the forthcoming lemma to be valid. One can simply set $\eta = 1/100$.

Lemma 1. *Let* l_1, l_2 *be two distinct members of the ratio set* $A/A \subset \mathbb{C}$, *with some realization* $l_1 = y_1/x_1$ *and* $l_2 = y_2/x_2$, *for* $x_1, y_1, x_2, y_2 \in A$. *Consider* l_1, l_2 *as points in* $\mathbb{R}^2$. *Then the point* $Z = (y_1 + y_2)/(x_1 + x_2)$ *considered as a point from* $\mathbb{R}^2$ *lies in some open set* $M_{(l_1,l_2)}$, *containing the open straight line interval* $(l_1, l_2) = \{tl_1 + (1-t)l_2 : t \in (0,1)\}$ *and* $M_{(l_1,l_2)}$ *is symmetric with respect to this line interval. Furthermore, consider a subset* R *of the ratio set as a vertex set of a tree* T *in* $\mathbb{R}^2$, *and let the sum of the Euclidean lengths of the edges of* T *be minimum, i.e., let* T *be a minimal spanning tree on the set* R. *Then, if* (l_1, l_2) *runs over the edges of* T *then the sets* $M_{(l_1,l_2)}$ *are pairwise disjoint.*

The above lemma represents a bona fide generalization of the construction of Solymosi [10] for the positive reals. The lemma proven in [8] was the crucial tool to establish the sum–product result for complex numbers.

To prove Theorem 1, we use induction on k. For $k = 0$ the assertion is obvious. Assuming that it is true for $k \geqslant 0$, we prove it for $k+1$. Notice that $u_{k+1} - u_k = (2^{k+2} - 1)/2 \geqslant 3/2$. Hence, for $K \geqslant n^{1/3}$ we get

$$N^{(1+(k+1)/2)} K^{-u_{k+1}} / \left(N^{(1+k/2)} K^{-u_k} \right) \leqslant N^{1/2} K^{-3/2} \leqslant 1,$$

and the result follows from the trivial inequality $|2^{k+1}A| \geqslant |2^k A|$. So, we will assume that

$$1 \leqslant K < n^{1/3}. \tag{2.1}$$

We recall that the multiplicative energy $E(A)$ is defined as the number of quadruples

$$(a_1, a_2, a_3, a_4) \in A \times A \times A \times A$$

with $a_1 a_2 = a_3 a_4$. For any $z \in AA$ we denote

$$N(z) = |\{(x, y) \in A \times A : \ z = xy\}|.$$

Since

$$\sum_{z \in AA} N(z) = n^2, \quad \sum_{z \in AA} N(z)^2 = E(A),$$

We conclude from the Cauchy–Schwartz inequality that

$$E(A) \geqslant n^4 / |AA| \geqslant n^3 K^{-1}.$$

Similarly, for any $l \in A/A$ we denote

$$M(l) = |\{(x, y) \in A \times A : \ l = y/x\}|.$$

We have

$$\sum_{l \in A/A} M(l) = n^2, \quad \sum_{l \in A/A} M(l)^2 = E(A) \geqslant n^3 K^{-1}. \tag{2.2}$$

Let $\mathcal{L}$ be the set of $l \in A/A$ with $M(l) \geqslant nK^{-1}/2$. We observe that

$$\sum_{l\in(A/A)\setminus\mathcal{L}} M(l)^2 \leqslant \sum_{l\in(A/A)\setminus\mathcal{L}} M(l)nK^{-1}/2 \leqslant n^3K^{-1}/2.$$

Therefore, by (2.2),

$$\sum_{l\in\mathcal{L}} M(l)^2 \geqslant n^3K^{-1}/2.$$

Taking into account that $M(l) \leqslant n$ for any l, we obtain that

$$|\mathcal{L}| \geqslant nK^{-1}/2. \tag{2.3}$$

Without loss of generality, we will assume that $n \geqslant 3$. Then, by (2.1),

$$nK^{-1}/2 > 1. \tag{2.4}$$

For any $l \in \mathcal{L}$ we consider the set

$$A_l = \{x \in A : lx \in A\}.$$

We have

$$|A_l| = M(l) \geqslant nK^{-1}/2.$$

Trivially, $|A_lA_l| \leqslant |AA|$. Therefore,

$$|A_lA_l| \leqslant K'|A_l|, \quad K' = 2K^2.$$

Applying the induction hypothesis to the set A_l we get

$$|2^kA_l| \geqslant C_1\left(nK^{-1}\right)^{(1+k/2)} K^{-2u_k} = C_1n^{1+k/2}K^{-2u_k-1-k/2} := M. \tag{2.5}$$

where $c_1 = C_1(k) > 0$.

Every $x \in 2^kA_l$ has a representation $x = \sum_{j=1}^{2^k} x_j$ with $x_j \in A_l$. Therefore,

$$lx = \sum_{j=1}^{2^k} lx_j.$$

We observe that, by definition of A_l, all numbers x_j and lx_j belong to A. Thus, $x, lx \in 2^k A$.

Now we consider two distinct $l_1, l_2 \in \mathcal{L}$. The equality

$$(x_1, l_1x_1) + (x_2, l_2x_2) = (x_1', l_1x_1') + (x_2', l_2x_2') \in \mathbb{C}^2$$

holds only if $x_1 = x_1'$, $x_2 = x_2'$. Thus, we get at least M^2 distinct points

$$(x, y) = (x_1 + x_2, l_1x_1 + l_2x_2) \in 2^{k+1}A \times 2^{k+1}A, \quad x_1 \in 2^k A_{l_1},\ x_2 \in 2^k A_{l_2}.$$

By Lemma 1 for all these x, y the ratio y/x considered as a point from $\mathbb{R}^2$ lies in the set $M_{(l_1,l_2)}$. Now let (l_1, l_2) run over the edges of the graph constructed in Lemma 1. Since the sets $M_{(l_1,l_2)}$ are disjoint, for distinct edges we get distinct pairs (x, y). The number V of edges is $|\mathcal{L}| - 1$. By (2.3) and (2.4),

$$V \geqslant nK^{-1}/4. \tag{2.6}$$

By (2.5) and (2.6), we have constructed at least M^2V pairs $(x,y)\in 2^{k+1}A\times 2^{k+1}A$. So, for some $C_2 = C_2(k) > 0$ we have

$$|2^{k+1}A| \geqslant MV^{1/2} \geqslant C_2 n^{3/2+k/2} K^{-2u_k-3/2-k/2}.$$

Observing that $2u_k + 3/2 + k/2 = u_{k+1}$ we conclude the proof of the theorem.

It is possible to improve the exponent of K using that we always can find a set $\mathcal{L}' \subset A/A$ such that the lower estimate either for its cardinality or for the numbers $|A_l|$ will be better than indicated above. However, such an improvement will not be very significant, and we did not try to do this job.

For $A \subset \mathbb{R}$ with small $|AA|$ Shkredov [9] established better lower estimates for $|A + A|$. This result allows to improve the term -4 in the exponent in the statement of Theorem 1.

After the paper has been accepted, the author has learned that recently A. Bush and E. Croot (Few products, many \$h\$-fold sums, arXiv:1409.7349v1) drastically improved Theorem A.

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