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Point sets with every triangle having a large angle

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Abstract: We shall consider the following geometric objects. Let $0 < \alpha < \pi$. Call $H \subset \mathbb{R}^d$ a P_α-set, if every triangle in H either has an angle larger than α or is degenerate. Similarly, call a sequence $(X_i)_{i=1}^n$, $X_i \in \mathbb{R}^d$ a P_α-sequence, if for every $1 \leq i < j < k \leq n$, the angle $X_i X_j X_k \angle$ is larger than α.
In the present paper, we prove various bounds on the sizes of the largest P_α-sets and P_α-sequences in point sets of given cardinality. We also propose several open problems.

Keywords: almost collinear points, Erdos problem, discrete geometry, large angles

AMS Subject classification: 52C10

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1. Introduction

A famous conjecture by Erdős claims that among any $2^d + 1$ points in the d-dimensional Euclidean space there are three forming an obtuse triangle. The conjecture was proven to be true by Danzer and Grünbaum [1].

Furthermore, Erdős proposed the following question. Let $m \geq 3$ and $d \geq 2$ be integers. What is the largest number $\gamma_d(m)$ such that any set of m points in the d-dimensional space contains a triangle with an angle at least $\gamma_d(m)$. Erdős and Szekeres [4] proved that if $2^t < m \leq 2^{t+1}$ with some positive integer t, then

$$\pi - \pi/t + \pi/t(2^t + 1)^2 \leq \gamma_2(m) \leq \pi - \pi/(t + 1).$$

In the present paper, we consider similar problems as the ones stated above for P_α-sets and P_α-series instead of triangles.

Let $0 < \alpha < \pi$.

DEFINITION 1.1. *A set $S \subset \mathbb{R}^d$ is a P_α-set, if every triangle formed by three distinct points of S either has an angle larger than α or is degenerate.*

DEFINITION 1.2. *A sequence $\{X_i\}_{i=1}^n$ in $\mathbb{R}^d$ is a P_α-sequence, if for every $1 \leq i < j < k \leq n$ the angle $X_iX_jX_k\angle$ is larger than α.*

We shall call $P_{\pi/2}$-sets and $P_{\pi/2}$-sequences *obtuse sets* and *obtuse sequences* respectively. Abusing notation we sometimes refer to a set of points as a P_α-sequence if the elements of the set can be ordered to form a P_α-sequence.

Our aim is to find bounds on the size of the largest P_α-sets and P_α-sequences in a point set of given cardinality. We note that if α is close to π, a P_α-sequence is a sequence of points which are "close to being collinear"; and if $\alpha \geq 2\pi/3$, then the points of a P_α-set can be ordered to form P_α-sequence as well, see Proposition 3.1.

Our paper is organized as follows. In the next section, we introduce the notion of *multi-posets*. The structure of multi-poset and its properties are some of the main tools used in our proofs. A set with multiple partial orders is a multi-poset if any two different elements of the set are comparable by at least one of the partial orders. This new structure poses some interesting problems itself, but in this paper we focus on its applications concerning P_α-sets and P_α-sequences.

In the third section, we prove various results about point sets in the plane. We shall give lower and upper bounds for the largest P_α-sets and P_α-sequences contained in point sets in the plane with given cardinality. We obtain different estimates for the cases $\alpha \geq \pi/2$ and $\alpha < \pi/2$ and we strengthen our results in the special cases $\alpha = \pi/3$ and $\alpha = \pi/2$. Also, we prove a sharp result on the minimum number of obtuse sets and sequences needed to decompose a point set of given cardinality.

In the fourth section, we investigate the largest P_α-set and P_α-sequence of a point set chosen randomly with uniform distribution in the square. The constructions provided in the third section having only small P_α-sets have very special structures, so it may be expected that a uniform random point set has much larger P_α-sets and P_α-sequences. We show that this is true.

In the fifth section, we generalize some of our results in higher dimensions. Finally, we finish our paper with some open problems.

A brief word about notion. Throughout this paper, if X, Y, Z are points in $\mathbb{R}^d$ the angle $XYZ\angle$ denotes the undirected angle of the triangle XYZ at vertex Y, so $XYZ\angle \in [0, \pi]$. However, if v, w are vectors in $\mathbb{R}^2$, then the angle from v to w means the directed angle taking its value modulo 2π and the absolute angle between v and w is the angle of the triangle with vertices $0, v, w$ at 0. Also, for points X, Y let $|XY|$ denote the distance between X and Y and let $[X, Y]$ denote the closed segment with endpoints X and Y. Finally, let $[n] = \{1, ..., n\}$ for any positive integer n.

2. Multi-posets

In this section we introduce the notion of multi-posets. We prove some properties of this structure that we shall use throughout this paper.

Definition 2.1. *Let $<_1, ..., <_r$ be partial orderings on the set H. Call*

$$(H, <_1, ..., <_r)$$

a multi-poset, if for any $x, y \in H$, $x \neq y$ there exists $1 \leq i \leq r$ such that $x <_i y$ or $y <_i x$.

Our first lemma is just a generalization of the well known Erdős-Szekeres [4] theorem: a sequence of $t^2 + 1$ real numbers contains a monotone subsequence with $t + 1$ elements.

Lemma 2.1. *Let t be a positive integer, $(H, <_1, ..., <_r)$ a multi-poset and $|H| \geq t^r + 1$. Then there exist $C \subset H$ and $1 \leq i \leq r$ such that C is a $<_i$-chain and $|C| \geq t + 1$.*

Also, if $|H| = \infty$ then there exist $C \subset H$ and $1 \leq i \leq r$ such that C is a $<_i$-chain and $|C| = \infty$.

Proof. We prove this by induction on r. If $r = 1$, then $(H, <_1)$ is a totally ordered set, so H itself is an $<_1$-chain and we are done. Suppose that $r > 1$ and the Lemma is true for $r - 1$ instead of r. Suppose that the longest $<_r$-chain has at most t elements (otherwise we are done), then by Mirsky's theorem [6] H is the union of at most t $<_r$-antichains. As $|H| \geq t^r + 1$ one of these antichains must contain at least $t^{r-1} + 1$ elements, denote this antichain by H'. But then $(H', <_1, ..., <_{r-1})$ is a multi-poset, so by induction there exist $C \subset H'$ and $1 \leq i \leq r$ such that C is a $<_i$-chain and $|C| \geq t + 1$.

If $|H| = \infty$, then the statement is an immediate consequence of the infinite Ramsey theorem. If $|H| = \infty$, then there exist $1 \le i \le r$ and $C \subset H$ such that $|C| = \infty$ and any two elements of C are comparable by $<_i$, thus C is an infinite $<_i$-chain. □

The previous lemma is sharp, in the next proposition we construct a multi-poset with t^r elements having no chain of length $t+1$.

PROPOSITION 2.1. *Let t be a positive integer. There exists a multi-poset $(H, <_1, ..., <_r)$ such that $|H| = t^r$ and for $i = 1, ..., r$ the longest $<_i$-chain has at most t elements.*

PROOF. Let $H = [t]^r$ and for $i = 1, ..., r$ define $<_i$ such that

$$(a_1, ..., a_r) <_i (b_1, ..., b_r)$$

if $a_i < b_i$. It is easy to see that $<_i$ is a partial ordering on H and any two elements of H are comparable, so $(H, <_1, ..., <_r)$ is a multi-poset. Furthermore, if $\mathbf{x}_1, ..., \mathbf{x}_k \in H$ form a $<_i$-chain, then the ith coordinates of $\mathbf{x}_1, ..., \mathbf{x}_k$ are pairwise different, so $k \le t$. Thus the longest $<_i$-chain has at most t elements. □

Next, we prove two lemmas about partitioning a multi-poset into chains.

LEMMA 2.2. *Let $t > 1$ be an integer and $(H, <_1, <_2)$ a multi-poset with $|H| \le t(t+1)/2 - 1$. Then H is the union of at most $t-1$ chains, meaning that there exist $C_1, ..., C_{t-1} \subset H$ such that C_i is a $<_1$-chain or a $<_2$-chain and $H = \bigcup_{i=1}^{t-1} C_i$.*

PROOF. We prove it by induction on t. If $t = 2$, then $|H| \le 2$ so H is a $<_1$-chain or a $<_2$-chain itself. Suppose $t > 2$ and the statement is true for $t-1$ instead of t.

If H contains a $<_1$-antichain with t elements, then let one such antichain be C_{t-1}. Then C_{t-1} is a $<_2$-chain and $|H \setminus C_{t-1}| \le t(t-1)/2 - 1$, so by the induction hypothesis $H \setminus C_{t-1} = \bigcup_{i=1}^{t-2} C_i$, where C_i is a $<_1$-chain or a $<_2$-chain. Thus $H = \bigcup_{i=1}^{t-1} C_i$ and we are done.

If the largest $<_1$-antichain in H has at most $t-1$ elements, then by applying Dilworth's theorem [2] we deduce that H is the union of at most $t-1$ pieces of $<_1$-chains. □

The above lemma is also sharp, one can easily construct a multi-poset with $t(t+1)/2$ elements and having no decomposition into less than t chains. We shall show an example for such a multi-poset later. A slightly weaker result can be proved for multi-posets with more than two relations.

LEMMA 2.3. *Let t, r be positive integers with $t \geq r$ and let $(H, <_1, \ldots, <_r)$ be a multi-poset with $|H| \leq \binom{t}{r} + 1$. Then H is the union of at most $\binom{t-1}{r-1}$ chains, meaning that there exist $C_1, \ldots, C_k \subset H$ such that $k \leq \binom{t-1}{r-1}$, $H = \bigcup_{i=1}^{k} C_i$ and for $i = 1, \ldots, k$ there exists $1 \leq j \leq r$ such that C_i is a $<_j$-chain.*

PROOF. We proceed by induction on r and t. If $r = 1$, then H is a totally ordered set and the statement trivially holds. If $t = r$, then $|H| \leq 2$, so H is a $<_i$-chain with some i. Suppose $r > 1$ and $t > r$ and the statement holds for $(t-1, r-1)$ and $(t, r-1)$ instead of (t, r).

First, suppose that H contains a $<_r$-antichain of size $\binom{t-1}{r-1}$. Let H' be such an antichain, then $(H', <_1, \ldots, <_{r-1})$ is a multi-poset. Applying the induction hypothesis, we have that H' is the union of at most $\binom{t-2}{r-2}$ chains. Furthermore,

$$|H \setminus H'| \leq \binom{t}{r} + 1 - \binom{t-1}{r-1} = \binom{t-1}{r} + 1,$$

so $H \setminus H'$ is the union of at most $\binom{t-2}{r-1}$ chains. Thus H is the union of at most $\binom{t-2}{r-2} + \binom{t-2}{r-1} = \binom{t-1}{r-1}$ chains.

It only remains to deal with the case, when the largest $<_r$-antichain has less than $\binom{t-1}{r-1}$ elements. Apply Dilworth's theorem [2] to get a chain decomposition of H into less than $\binom{t-1}{r-1}$ pieces of $<_r$-chains. □

3. P_α-sets and P_α-sequences in the plane

Szekeres [7] proved that among $2^n + 1$ points in the plane there are always three forming an angle larger than $\pi - \pi/n$. Furthermore, he showed that this statement is sharp in the following sense: for any $\epsilon > 0$ there exist 2^n points in the plane such that any angle formed by three of those points is less than $\pi - \pi/n + \epsilon$. In this section we generalize these results to P_α-sets and P_α-sequences.

THEOREM 3.1. *Let n be a positive integer and $0 < \alpha \leq (n-1)\pi/n$. Then any set $H \subset \mathbb{R}^2$, $|H| \geq t^n + 1$ contains a P_α-sequence with at least $t+1$ elements.*

Also, if $|H| = \infty$, then H contains an infinite P_α-sequence.

PROOF. If H is uncountable, replace H with any of its countable subsets. Take any vector v such that for all $A, B \in H$, $A \neq B$ the angle from v to $\overrightarrow{AB}$ is not a rational multiple of π (as H is countable or finite, there exists such a v). For $k = 1, ..., n$ define the partial ordering $<_k$ on H as follows: let $A <_k B$ if the angle from v to $\overrightarrow{AB}$ is between $(k-1)\pi/n$ and $k\pi/n$. It is easy to check that $(H, <_1, ..., <_n)$ is a multi-poset and a $<_k$-chain is a $P_{(n-1)\pi/n}$-sequence. Now apply Lemma 2.1 to deduce the result. □

THEOREM 3.2. *Let $n \geq 2$ be an integer and $\pi > \alpha > (n-1)\pi/n$. There exists a set $H \subset \mathbb{R}^2$ with $|H| = t^n$ such that the largest P_α-set of H has t elements.*

PROOF. Let ϵ, l be reals such that $0 < \epsilon < \alpha - (n-1)\pi/n$ and $l > 8tn/\epsilon$. Identify the plane with the complex numbers and let $\xi = \cos(\pi/n) + i\sin(\pi/n)$. For every $\mathbf{a} = (a_1, ..., a_n) \in [t]^n$ define the complex number

$$z_{\mathbf{a}} = \sum_{j=1}^{n} a_j l^j \xi^j.$$

Let $\mathbf{a}, \mathbf{b} \in [t]^n$ different, with ith coordinates a_i, b_i respectively. Let m be the largest index such that $a_m \neq b_m$, and suppose that $a_m > b_m$. We show that

$$\frac{m\pi}{n} - \frac{\epsilon}{2} < \arg(z_{\mathbf{a}} - z_{\mathbf{b}}) < \frac{m\pi}{n} + \frac{\epsilon}{2}.$$

Here,

$$z_{\mathbf{a}} - z_{\mathbf{b}} = \sum_{j=1}^{m} (a_j - b_j) l^j \xi^j =$$

$$= (a_m - b_m) l^m \left(\xi^m + \sum_{j=1}^{m-1} \frac{a_j - b_j}{a_m - b_m} l^{j-m} \xi^j \right),$$

thus we have

$$\left|\frac{z_{\mathbf{a}}-z_{\mathbf{b}}}{l^m(a_m-b_m)}-\xi^m\right| = \left|\sum_{j=1}^{m-1}\frac{a_j-b_j}{a_m-b_m}l^{j-m}\xi^j\right| \leq$$

$$\leq \sum_{j=1}^{m-1}\left|\frac{a_j-b_j}{a_m-b_m}\right| l^{j-m} < \frac{nt}{l} < \frac{\epsilon}{8}.$$

This means that the distance of the complex number $(z_{\mathbf{a}}-z_{\mathbf{b}})/l^m(a_m-b_m)$ and the root of unity ξ^m is less than $\epsilon/8$, so their arguments differ in at most $\arcsin \epsilon/8 < \epsilon/2$ (using that $0<\epsilon<\pi$). This proves that

$$\frac{m\pi}{n}-\frac{\epsilon}{2} < \arg(z_{\mathbf{a}}-z_{\mathbf{b}}) < \frac{m\pi}{n}+\frac{\epsilon}{2}.$$

Now we show that the set $H=\{z_{\mathbf{a}} : \mathbf{a}\in[t]^n\}$ does not contain a P_α-set with $t+1$ elements. It suffices to show that for an arbitrary $I\subset[t]^n$ with $|I|=t+1$, the set $S=\{z_{\mathbf{a}} : \mathbf{a}\in I\}$ contains a triangle, whose every angle is smaller than α.

Let k be the largest index such that the kth coordinate of the elements of I are not all the same. As I has $t+1$ elements, there are two amongst them whose kth coordinate is the same. Let $\mathbf{a},\mathbf{b}\in I$ be such elements and suppose that the largest index they differ in is m. Finally, choose $\mathbf{c}\in I$ such that the kth coordinate of $\mathbf{c}$ differs from the kth coordinate of $\mathbf{a},\mathbf{b}$. Then every angle of the triangle $z_{\mathbf{a}}, z_{\mathbf{b}}, z_{\mathbf{c}}$ is less than α:

$$\arg(z_{\mathbf{a}}-z_{\mathbf{b}}) \in \left(\frac{m\pi}{n}-\frac{\epsilon}{2},\frac{m\pi}{n}+\frac{\epsilon}{2}\right)\cup\left(\pi+\frac{m\pi}{n}-\frac{\epsilon}{2},\pi+\frac{m\pi}{n}+\frac{\epsilon}{2}\right)$$
$$\arg(z_{\mathbf{a}}-z_{\mathbf{c}}) \in \left(\frac{k\pi}{n}-\frac{\epsilon}{2},\frac{k\pi}{n}+\frac{\epsilon}{2}\right)\cup\left(\pi+\frac{k\pi}{n}-\frac{\epsilon}{2},\pi+\frac{k\pi}{n}+\frac{\epsilon}{2}\right)$$

thus the angle of the triangle $z_{\mathbf{a}}, z_{\mathbf{b}}, z_{\mathbf{c}}$ at $z_{\mathbf{a}}$ is either in

$$\left(\frac{(k-m)\pi}{n}-\epsilon,\frac{(k-m)\pi}{n}+\epsilon\right)$$

or in

$$\left(\frac{(n-(k-m))\pi}{n}-\epsilon,\frac{(n-(k-m))\pi}{n}+\epsilon\right).$$

Hence this angle is smaller than α but larger than $\pi - \alpha$, meaning that the other two angles of the triangle must be also smaller than α. $\square$

If $\alpha < \pi/3$, then every set of points is a P_α-set, but not necessarily can be ordered to form a P_α-sequence. If $\alpha \leq \pi/2$, every set with N points contains a P_α-sequence with at least $\sqrt{N}$ points, however the next theorem shows that for every $\alpha > 0$ there are sets containing no P_α-sequence of length more than $c_\alpha\sqrt{N}$.

THEOREM 3.3. *For every $\alpha > 0$ there exists a constant c_α such that the $t \times t$ square grid does not contain a P_α-sequence with more than $c_\alpha t$ elements.*

PROOF. Let $H = [t]^2$ denote the $t \times t$ square grid in $\mathbb{R}^2$. For any 2-dimensional vector v with integer coordinates define the partial ordering $<_v$ in the following way: for $X, Y \in H$ let $X <_v Y$ if $\overrightarrow{XY} = \lambda v$ with some $\lambda > 0$.

First, we show that if $v = (x, y)$, then H is the union of at most $(|x| + |y|)t$ pieces of $<_v$-chains. For simplicity, we suppose that $x, y \geq 0$, as the other cases are similar. Let C be a maximal $<_v$-chain, then the minimal element of C has the form (a, b), where $a \leq x$ or $b \leq y$, otherwise $(a - x, b - y) \in H$ is smaller than every element of C. Furthermore, the maximal $<_v$-chains partition H and there are at most $(|x| + |y|)t$ elements in H, whose first coordinate is at most x or second coordinate is at most y. So there are at most $(|x| + |y|)t$ maximal chains.

Let $n = 2\lceil 2\pi/\alpha \rceil$ and choose vectors $v_1, ..., v_n$ such that for $i = 1, ..., n$ the angle from the x-axis to v_i is between $2(i-1)\pi/n$ and $2i\pi/n$ and v_i has integer coordinates. One can obviously choose such a vector with rational coordinates and a scalar multiple of this vector suffices.

Suppose $v_i = (x_i, y_i)$. Now we show that setting $c_\alpha = 1 + \sum_{i=1}^{n}(|x_i| + |y_i|)$ guarantees that H does not contain a P_α-sequence with more than $c_\alpha t$ elements.

Conversely, suppose that $X_1, ..., X_k$ is a P_α-sequence, where $k > c_\alpha t$. Let $S = \{X_1, ..., X_k\}$. Call $A \in S$ good, if for every $i = 1, ..., n$ there exists $B \in S$ such that $A <_{v_i} B$ and call A bad otherwise. If an element of S is bad, it is the maximal element of some maximal $<_{v_i}$-chain in S. But S is the union of at most $(|x_i| + |y_i|)t$ maximal $<_{v_i}$-chains, thus the number of bad elements is at most $\sum_{i=1}^{n}(|x_i| + |y_i|)t$. By the choice of c_α there is at least one good element in S, suppose it is X_m. Choose elements $X_{i_1}, ..., X_{i_n} \in S$ such that $X_m <_{v_j} X_{i_j}$. The angle $X_{i_j}X_mX_{i_{j+1}}\angle$ is equal to the absolute angle between v_j and v_{j+1}, which is smaller than $4\pi/n \le \alpha$. (Here $j = 1, ..., n$ and indices meant modulo n.) Hence by the definition of the P_α-sequence i_j and i_{j+1} are both smaller or both larger than m. This implies that $i_1, ..., i_n$ are all smaller or all larger than m. However, look at the triangle $X_{i_1}X_{i_{1+n/2}}X_m$. As $X_1, ..., X_n$ is a P_α-sequence, the angle at X_{i_1} or the angle at $X_{i_{1+n/2}}$ is larger than α. But the angle at X_m is equal to the absolute angle between v_1 and $v_{1+n/2}$, which is at least $\pi - 4\pi/n > \pi - \alpha$, so the other two angles must be smaller than α. We arrived to a contradiction. □

It is possible to take v_i such that $|x_i| + |y_i| < 4n$, so one can deduce from the above proof that the minimal c_α is $O(\alpha^{-2})$.

Finally, we prove that in a point set with N elements there is always a $P_{\pi/3}$-sequence of size at least $2\sqrt{N}/\sqrt{3}$, which is stronger than the corresponding bound of Theorem 3.1.

THEOREM 3.4. *Let $H \subset \mathbb{R}^2$ with $|H| > 3t^2 - 3t + 1$. Then H contains a $P_{\pi/3}$-sequence with at least $2t$ elements.*

PROOF. Take any direction v such that for all $X, Y \in H$, $X \neq Y$ the angle from v to $\overrightarrow{XY}$ is not a rational multiple of π. Define the partial orderings $<_1, <_2, <_3$ on $\mathbb{R}^2$ as follows: for $i = 1, 2, 3$ let $X <_i Y$ if the angle from v to $\overrightarrow{XY}$ is between $(i-1)\pi/3$ and $i\pi/3$. It is easy to check that $(H, <_1, <_2, <_3)$ is a multi-poset.

Observe that if $A \subset H$ is a $<_i$-antichain, then the elements of A can be ordered to form a $P_{\pi/3}$-sequence. We show this for $i = 1$, the other cases are similar. Define the relation $<_{2,3}$ as follows: let $X <_{2,3} Y$ if $X <_2 Y$ or $X <_3 Y$. Then $<_{2,3}$ is also a partial ordering on H (using the geometry of $<_2$ and $<_3$), and a $<_1$-antichain is a $<_{2,3}$-chain, whose elements can be ordered to form a $P_{\pi/3}$-sequence. Thus it is enough to show that H contains a $<_i$-antichain with $2t$ elements for some $i \in \{1, 2, 3\}$.

For $i = 1, 2, 3$ let

$$v_i = \left(\cos\frac{(i-1/2)\pi}{3}, \sin\frac{(i-1/2)\pi}{3}\right).$$

Let C be a $<_i$-chain in H with elements $X_1 <_i ... <_i X_s$. Define the broken line $L(C)$ as follows: connect X_j and X_{j+1} with a segment if $j = 1, ..., s-1$ and draw a half line from X_1 to the direction of $-v_i$ and a half line from X_s to the direction v_i (L is dependent of i, but for simplicity we shall not mark this dependance). It is easy to see that $L(C)$ is a $<_i$-chain too.

Before we proceed, we need the following lemma.

LEMMA 3.4. *Let $i \in \{1, 2, 3\}$ and let $C_1, ..., C_k$ be finite $<_i$-chains. Then there exist $<_i$-chains $C'_1, ..., C'_k$ such that $\bigcup_{j=1}^{k} C_j = \bigcup_{j=1}^{k} C'_j$ and the broken lines $L(C'_1), ..., L(C'_k)$ are pairwise disjoint.*

PROOF OF LEMMA. Suppose that no three points in the union are collinear, otherwise adding a small noise is not changing the set of $P_{\pi/3}$-sequences. It can be also assumed that $C_1, ..., C_k$ are pairwise disjoint. Let S be a square covering $\bigcup_{j=1}^{k} C_j$ and having two sides parallel to v_i. Let $L_0(C) = L(C) \cap S$ for any chain C. Then $L_0(C)$ has a finite length, denote this length by $l(C)$. Also, $L(C_1) \setminus L_0(C_1), ..., L(C_k) \setminus L_0(C_k)$ are unions of parallel half lines, so every intersection of $L(C_1), ..., L(C_k)$ is inside S.

Suppose that for some a and b the broken lines $L_0(C_a)$ and $L_0(C_b)$ intersect. Let $X_1 <_i ... <_i X_m$ be the points of C_a and $Y_1 <_i ... <_i Y_n$ be the points of C_b. Choose one intersection of $L_0(C_a)$ and $L_0(C_b)$. If $[X_j, X_{j+1}]$ and $[Y_{j'}, Y_{j'+1}]$ intersect, let $C_a^* = \{X_1, ..., X_j, Y_{j'+1}, ..., Y_n\}$ and $C_b^* = \{Y_1, ..., Y_{j'}, X_{j+1}, ..., X_m\}$. (See Figure 1.) Then C_a^* and C_b^* are $<_i$-chains. Indeed, let Z be the intersection of $[X_j, X_{j+1}]$ and $[Y_{j'}, Y_{j'+1}]$, then $Y_{j'} <_i Z$ and $Z <_i X_{j+1}$ so $Y_{j'} <_i X_{j+1}$. Similarly, $X_j <_i Y_{j'+1}$.

Also $l(C_a) + l(C_b) > l(C_a^*) + l(C_b^*)$ holds. This is true as

$$\begin{aligned} l(C_a)+l(C_b)-(l(C_a^*)+l(C_b^*)) &= |X_{j+1}X_j|+|Y_{j'+1}Y_{j'}|-|X_jY_{j'+1}|-|X_jY_{j'+1}| = \\ &= |X_{j+1}Z| + |ZX_j| + |Y_{j'+1}Z| + |ZY_{j'}| - |X_jY_{j'+1}| - |Y_{j'}X_{j+1}| = \\ &= (|X_{j+1}Z| + |ZY_{j'}| - |Y_{j'}X_{j+1}|) + (|ZX_j| + |Y_{j'+1}Z| - |X_jY_{j'+1}|) > 0, \end{aligned}$$

where the last inequality is a consequence of the triangle inequality.

If $[X_j, X_{j+1}]$ intersects the half line with endpoint Y_1, let their intersection be Z and let $C_a^* = \{X_1, ..., X_j, Y_1, ..., Y_n\}$ and $C_b^* = \{X_{j+1}, ..., X_m\}$. Then C_a^* and C_b^* are also $<_i$-chains and $l(C_a) + l(C_b) > l(C_a^*) + l(C_b^*)$. To prove this, first define d_1, d_2, d_3 as follows. Let u be the side of S which intersects the half line of $L(C_a)$ from Y_1. Then let d_1, d_2, d_3 be the distances of the points Y_1, X_{j+1}, Z from

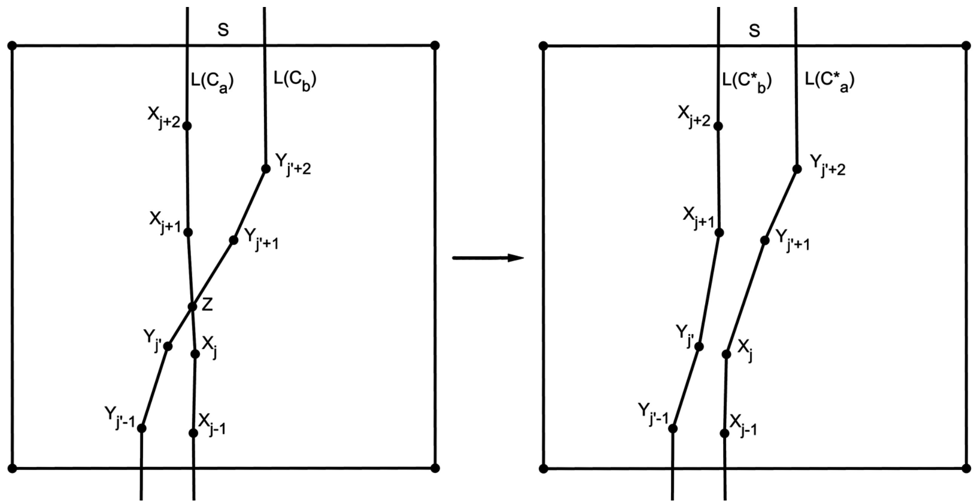

Fig. 1. A step of the procedure

u respectively. Then

$$l(C_a) + l(C_b) - l(C_a^*) - l(C_b^*) = |X_jX_{j+1}| + d_1 - d_2 - |X_jY_1| =$$
$$= |X_jZ| + |ZX_{j+1}| + d_3 + |Y_1Z| - d_2 - |X_jY_1| =$$
$$= (|X_jZ| + |ZY_1| - |X_jY_1|) + |ZX_{j+1}| + d_3 - d_2 > |ZX_{j+1}| + d_3 - d_2 > 0,$$

where the last inequality holds as $d_3 - d_2$ is the length of the orthogonal projection of $\overrightarrow{X_{j+1}Z}$ to the line with direction v.

The cases when some $[X_j, X_{j+1}]$ intersects the half line with endpoint Y_n or some $[Y_j, Y_{j+1}]$ intersects one of the half lines of $L(C_a)$ can be handled as the previous case.

Repeat the following procedure: if there exist a and b integers such that $1 \leq a < b \leq k$ and the broken lines $L(C_a)$ and $L(C_b)$ intersect, then replace C_a and C_b with C_a^* and C_b^*; otherwise stop. After each step of the procedure $\bigcup_{j=1}^{k} C_j$ remains the same, as $C_a^* \cup C_b^* = C_a \cup C_b$. Also, $\sum_{j=1}^{k} l(C_k)$ strictly decreases, as $l(C_a^*) + l(C_b^*) < l(C_a) + l(C_b)$. There are only finitely many ways to select k chains

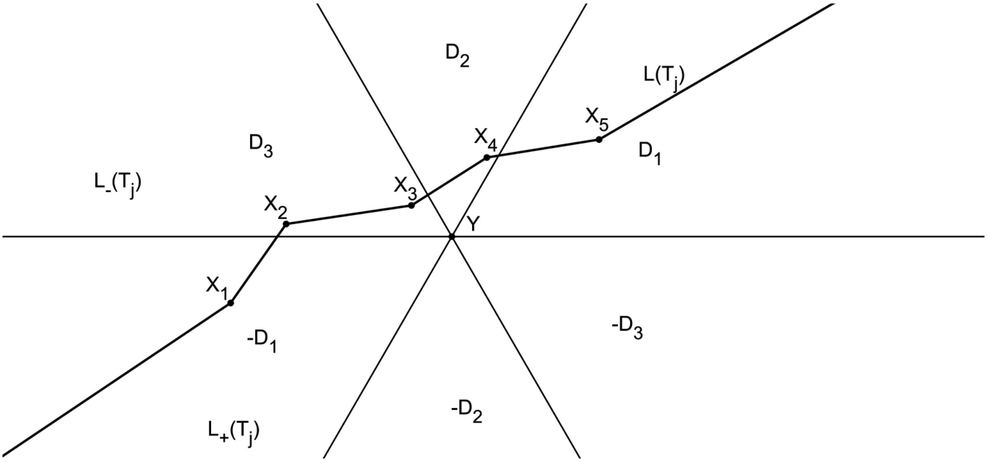

Fig. 2. $L_+(T_j)$ and $L_-(T_j)$

from the finite set $\bigcup_{j=1}^{k} C_j$, hence the procedure must stop after finitely many steps. When the procedure stops, it produces chains $C'_1, \ldots, C'_k$, where $\bigcup_{j=1}^{k} C_j = \bigcup_{j=1}^{k} C'_j$ and $L(C'_1), \ldots, L(C'_k)$ are pairwise disjoint. □

Suppose that for $i = 1, 2, 3$ the largest $<_i$-antichain has size at most $2t-1$. Then by Dilworth's theorem H is the union of $2t-1$ pieces of $<_i$-chains. Let $T_1, \ldots, T_{2t-1}$ be $<_1$-chains such that $\bigcup_{j=1}^{2t-1} T_j = H$ and let $U_1, \ldots, U_{2t-1}$ be $<_2$-chains such that $\bigcup_{j=1}^{2t-1} U_j = H$. Then by the previous lemma we can suppose that $L(T_1), \ldots, L(T_{2t-1})$ are pairwise disjoint and $L(U_1), \ldots, L(U_{2t-1})$ are pairwise disjoint as well.

For $j = 1, \ldots, 2t-1$ the broken line $L(T_j)$ divides the plane into two parts. Define these parts $L_+(T_j)$ and $L_-(T_j)$ as follows: let $Y \in \mathbb{R}^2 \setminus L(T_j)$ and let $D_i = \{X : Y <_i X\}$ and $-D_i = \{X : X <_i Y\}$. As $L(T_j)$ is a $<_1$-chain, it can only intersect one of D_2 and $-D_2$, and one of D_3 and $-D_3$. Also, $L(T_j)$ is continuous, $Y \notin L(T_j)$ and it contains points of D_1 and $-D_1$, hence it intersects exactly one of $D_2 \cup D_3$ and $(-D_2) \cup (-D_3)$. If it intersects $D_2 \cup D_3$, let $Y \in L_+(T_j)$, otherwise let $Y \in L_-(T_j)$.

The broken lines $L(T_1),\dots,L(T_{2t-1})$ are pairwise disjoint, so if $j\neq l$ then $L_+(T_j)\subset L_+(T_l)$ or $L_+(T_l)\subset L_+(T_j)$. Without the loss of generality

$$L_+(T_1)\subset\dots\subset L_+(T_{2t-1}).$$

Similarly, we can define $L_+(U_j)$ and $L_-(U_j)$ with D_3 instead of D_2 and $-D_1$ instead of D_3. Again, without the loss of generality let

$$L_+(U_1)\subset\dots\subset L_+(U_{2t-1}).$$

For every $1\leq j,l\leq 2t-1$ the broken lines $L(T_j)$ and $L(U_l)$ intersect, let their intersection be $Z_{j,l}$. Note that $H\subset\{(Z_{j,l}\,|\,j,l=1,\dots,2t-1\}$. We shall show that if $1\leq j<l\leq 2t-1$ and $1\leq k\leq 2t-1$, then $Z_{j,k}<_2 Z_{l,k}$. Indeed, as $Z_{j,k},Z_{l,k}\in L(U_k)$, we have that $Z_{j,k}<_2 Z_{l,k}$ or $Z_{l,k}<_2 Z_{j,k}$. But $Z_{j,k}\in L(T_j)\subset L_+(T_l)$, so only $Z_{j,k}<_2 Z_{l,k}$ can hold. Similarly, $Z_{k,l}<_1 Z_{k,j}$.

Next, we show that if $Z_{a,b}<_3 Z_{c,d}$ for some $1\leq a,b,c,d\leq 2t-1$, then $a<c$ and $b<d$. If $a=c$, then $Z_{a,b}<_1 Z_{c,d}$ or $Z_{c,d}<_1 Z_{a,b}$, so it is impossible. Suppose that $a>c$. Then $Z_{c,d}\in L(T_c)\subset L_+(T_a)$, so according to the definition of $L_+(T_a)$ one of the following four relations holds: $Z_{a,b}<_1 Z_{c,d}$, $Z_{c,d}<_1 Z_{a,b}$, $Z_{c,d}<_2 Z_{a,b}$ or $Z_{c,d}<_3 Z_{a,b}$. Hence $a\geq c$ cannot be true. One can show similarly that $b<d$ holds as well.

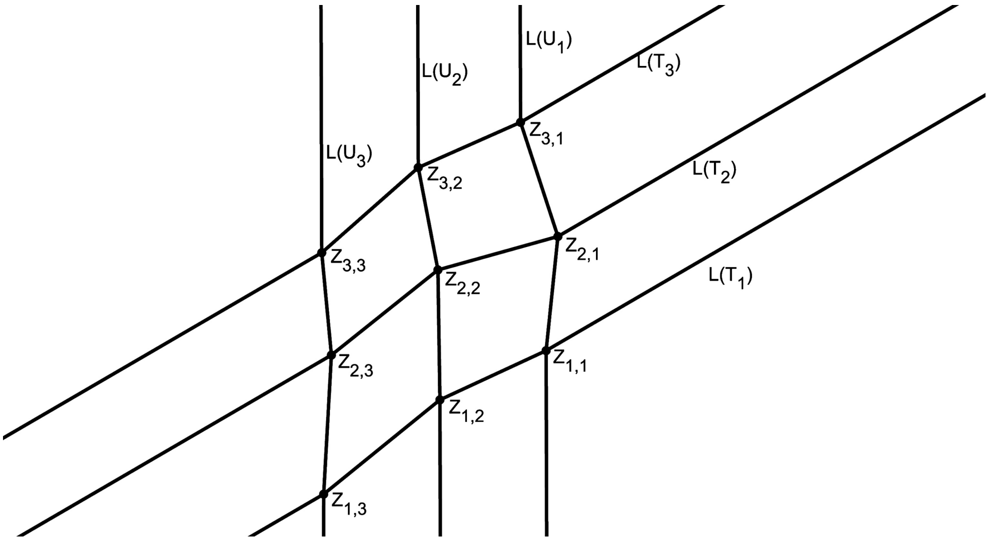

Fig. 3. The grid like structure of the $Z_{i,j}$'s

We also need the following lemma.

LEMMA 3.5. *Let* n *be a positive integer and* $S=\{(a,b)\,|\,a,b=1,\dots,n\}$. *Define the relation* $\prec$ *on* S *such that* $(a,b)\prec(c,d)$ *if* $a<c$ *and* $b<d$. *Then clearly* $(S,\prec)$ *is a poset. Let* $C_1,\dots,C_k$ *be* $\prec$*-chains* $(k\in\mathbb{N})$. *If* k *is odd, then*

$$\left|\bigcup_{i=1}^{k} C_i\right| \le kn-\frac{k^2-1}{4}$$

and if k *is even, then*

$$\left|\bigcup_{i=1}^{k} C_i\right| \le kn-\frac{k^2}{4}.$$

PROOF OF LEMMA. For $i=1,\dots,n$ let

$$A_i=\{(i,j)\,|\,j=i,\dots,n\}\cup\{(j,i)\,|\,j=i,\dots,n\}.$$

Then $|A_i|=2i-1$ and the disjoint union of $A_1,\dots,A_n$ is S. Furthermore, A_i is a $\prec$-antichain, which means that for $l=1,\dots,k$ the intersection of C_l and A_i contains at most one point. Hence

$$\left|A_i\cap\bigcup_{l=1}^{k} C_l\right| \le \min\{|A_i|,k\}=\min\{2i-1,k\}.$$

But then

$$\left|\bigcup_{l=1}^{k} C_l\right| = \left|\left(\bigcup_{i=1}^{n} A_i\right)\cap\left(\bigcup_{l=1}^{k} C_l\right)\right| \le$$

$$\le \sum_{i=1}^{n}\left|A_i\cap\left(\bigcup_{l=1}^{k} C_l\right)\right| \le \sum_{i=1}^{n}\min\{2i-1,k\}.$$

If k is odd, then

$$\sum_{i=1}^{n}\min\{2i-1,k\}=kn-\frac{k^2-1}{4},$$

and if k is even, then

$$\sum_{i=1}^{n}\min\{2i-1,k\}=kn-\frac{k^2}{4}. \qquad \square$$

Let $C_1,\ldots,C_{2t-1}$ be $<_3$-chains, whose union is H. Define the relation $\prec$ on $\{Z_{j,l}\}$ as follows: $Z_{a,b}\prec Z_{c,d}$ if $a<c$ and $b<d$. Then $<_3\subset\prec$ (which means $X<_3 Y\Rightarrow X\prec Y$), so $C_1,\ldots,C_{2t-1}$ are $\prec$-chains too. Let

$$S=\{(a,b)\,|\,a,b=1,\ldots,2t-1\}$$

and let $\phi:H\to S$ be an injection defined by $\phi(Z_{j,l})=(j,l)$ for $Z_{j,l}\in H$. Then

$$\phi:(H,\prec)\to(\phi(H),\prec)$$

is an isomorphism, where $(S,\prec)$ is defined as in the lemma above. Applying the lemma with parameters $n=2t-1$, $k=2t-1$ and $\prec$-chains $\phi(C_1),\ldots,\phi(C_{2t-1})$, we have that

$$\left|\bigcup_{i=1}^{2t-1}\phi(C_i)\right|\leq(2t-1)(2t-1)-\frac{(2t-1)^2-1}{4}=3t^2-3t+1<|H|.$$

Hence the union of $C_1,\ldots,C_{2t-1}$ cannot be H, which is a contradiction. This finishes the proof of the theorem. □

We finish this subsection by showing that for $\alpha\geq 2\pi/3$, the elements of a P_α-set can be ordered to form a P_α-sequence.

Proposition 3.1. *Let $2\pi/3\leq\alpha<\pi$. If $H\subset\mathbb{R}^2$ is a P_α-set with N points, then the elements of H have an enumeration $X_1,\ldots,X_N$ such that $\{X_i\}_{i=1}^N$ is a P_α-sequence.*

Proof. Let $A,B\in H$ such that $|AB|=\max_{U,V\in H}|UV|$. Let $X_1,X_2,\ldots,X_N$ be an enumeration of the points of H such that $X_1=A$, $X_N=B$ and $|AX_2|\leq|AX_3|\leq$ $\leq\ldots|AX_N|$. We show that $\{X_i\}_{i=1}^N$ is a P_α-sequence.

Let $1<i<N$. As AB is the largest side of the triangle ABX_i, the largest angle of this triangle is at X_i, so $AX_iB\angle>\alpha$ as H is a P_α-set.

Let $1<i<j\leq N$. As $BAX_i\angle<\pi-\alpha\leq\pi/3$ and similarly, $BAX_j\angle<\pi/3$, we have that $X_iAX_j\angle<2\pi/3$. So either $AX_iX_j\angle>\alpha$ or $AX_jX_i\angle>\alpha$. But $|AX_i|\leq|AX_j|$, so it must be $AX_iX_j\angle>\alpha$.

Let $1<i<j<k\leq N$. As $AX_iX_j\angle>\alpha$, we must have $AX_jX_i\angle<\pi-\alpha\leq\pi/3$. Also, $AX_jX_k\angle>\alpha$. Considering all the possible configurations of the points X_i,X_j,X_k, the angle $X_iX_jX_k\angle$ is equal to one of the following three values:

$AX_jX_i\angle + AX_jX_k\angle$; $AX_jX_k\angle - AX_jX_i\angle$; $2\pi - (AX_jX_i\angle + AX_jX_k\angle)$. In either case, $X_iX_jX_k\angle$ is at least $\pi/3$, which forces $X_iX_jX_k\angle > \alpha$ as H is a P_α-set and $\alpha \geq 2\pi/3$. □

This proposition is true for a d-dimensional space as well, the same proof can be repeated with slight modifications.

3.1. Obtuse sets

In this section, we shall strengthen our results in the case $\alpha = \pi/2$. It is a well known problem that amongst any five points in the plane there are three points, which form an obtuse angle, and it is sharp as if four points form a rectangle, there is no obtuse angle.

In the case of obtuse subsets, it is harder to find a set H of the largest cardinality such that H does not contain an obtuse set with t elements. The example of the rectangle suggests that maybe a $u \times u$ square grid is optimal with some u, however it is not hard to show that its largest obtuse set has $2u-2$ elements. The following theorem shows that if $t \geq 6$, the square grid can be beaten.

THEOREM 3.5. *There exists a set $H \subset \mathbb{R}^2$ such that $|H| = t(t+3)/2 - 6$ and the largest obtuse subset of H has t points.*

PROOF. For $u = 4, ..., t, t+1$ define the set of points

$$H_u = \left\{ \left(20^{u^2} \cos \frac{\pi(j-1)}{10 \cdot 2^u}, 20^{u^2} \sin \frac{\pi(j-1)}{10 \cdot 2^u} \right) : j = 0, ..., u-1 \right\}$$

and delete an arbitrary point from H_{t+1}. Let $H = \bigcup_{u=4}^{t+1} H_u$. First of all, $|H_u| = u$ if $4 \leq u \leq t$, $|H_{t+1}| = t$ and the sets H_u are pairwise disjoint. Hence

$$|H| = t(t+3)/2 - 6.$$

Clearly, the points of H_u lie on a circle of radius $(20)^{u^2}$ and center 0. We show that H has the following properties:

(i) Let $t+1 \geq a > b \geq 4$ and $A, B \in H_a$, $C \in H_b$. Then the triangle ABC is acute.

(ii) If $5 \leq a \leq t+1$, then for every $A \in H_a$ there exist $B, C \in H_{a-1}$, $B \neq C$ such that the triangle ABC is acute.

(i) The set of points X for which the angles $ABX\angle$ and $BAX\angle$ are acute is an open strip, whose borders are the two parallel lines going through A and B and orthogonal to AB. As A and B lie on a circle with center 0, the strip contains a circle with radius

$$\frac{|AB|}{2} \geq 20^{a^2} \sin \frac{\pi}{10 \cdot 2^a} > 20^{a^2} \frac{1}{10 \cdot 2^a} > 20^{a^2-a} > 20^{b^2}.$$

Thus the strip contains H_b, hence it contains C. It only remains to check that the angle $ACB\angle$ is acute as well. If it is not acute, then AB should be the largest side of the triangle, but

$$|AB| \leq 2 \cdot 20^{a^2} \sin \frac{\pi}{10} < \frac{2\pi}{10} 20^{a^2} < 20^{a^2} - 20^{b^2} \leq |AC|.$$

(ii) Suppose $A = \left(20^{a^2} \cos \frac{\pi(j-1)}{10 \cdot 2^a}, 20^{a^2} \sin \frac{\pi(j-1)}{10 \cdot 2^a}\right)$ for some $0 \leq j \leq a-1$. If j is odd, let

$$B = \left(20^{(a-1)^2} \cos \frac{\pi(\frac{j-1}{2} - 1)}{10 \cdot 2^{a-1}}, 20^{(a-1)^2} \sin \frac{\pi(\frac{j-1}{2} - 1)}{10 \cdot 2^{a-1}}\right),$$

$$C = \left(20^{(a-1)^2} \cos \frac{\pi \frac{j+1}{2}}{10 \cdot 2^{a-1}}, 20^{(a-1)^2} \sin \frac{\pi \frac{j+1}{2}}{10 \cdot 2^{a-1}}\right),$$

and if j is even, then

$$B = \left(20^{(a-1)^2} \cos \frac{\pi(\frac{j}{2} - 1)}{10 \cdot 2^{a-1}}, 20^{(a-1)^2} \sin \frac{\pi(\frac{j}{2} - 1)}{10 \cdot 2^{a-1}}\right),$$

$$C = \left(20^{(a-1)^2} \cos \frac{\pi \frac{j}{2}}{10 \cdot 2^{a-1}}, 20^{(a-1)^2} \sin \frac{\pi \frac{j}{2}}{10 \cdot 2^{a-1}}\right).$$

Since $a \geq 5$, we obtain that $B, C \in H_{a-1}$. Furthermore, A lies on the perpendicular bisector of B, C, so ABC is an acute triangle unless $BAC\angle \geq \pi/2$. But then BC is the largest side of the triangle, which is impossible as

$$|BC| < 2 \cdot 20^{(a-1)^2} < 20^{a^2} - 20^{(a-1)^2} \leq |AB|.$$

Let $S \subset H$ be obtuse. Let w be minimal such that H_w intersects S. Then $|S \cap H_u| \leq 1$ for $u > w$, using property (i). If $H_w \not\subset S$, then

$$|S| = \sum_{i=4}^{t+1} |S \cap H_i| \leq |H_w| - 1 + (t + 1 - w) = t.$$

If $w < t + 1$ and $H_w \subset S$, then by (ii) S cannot intersect H_{w+1}, so $|S| \leq |H_w| + (t - w) = t$. Finally, if $w = t + 1$, then $|S| \leq |H_{t+1}| = t$. □

Next, we prove a result about partitioning a point set into obtuse sets. The proof of the next theorem also presents an example of a set for which Lemma 2.2 is sharp.

THEOREM 3.6. *Let $t \geq 2$ be an integer and H a subset of the plane, $|H| \leq t(t+1)/2 - 1$. Then H can be partitioned into at most $t - 1$ obtuse sequences. This is sharp, there exists a set $S \subset \mathbb{R}^2$ such that $|S| = t(t + 1)/2$ and S cannot be partitioned into less than t obtuse sets.*

PROOF. Define the same multi-poset $(H, <_1, <_2)$ as in Theorem 3.1 for $n = 2$. Apply Lemma 2.2 for $(H, <_1, <_2)$ to deduce the first part of the theorem.

Now we construct a set S which satisfies the conditions. For $u = 1, ..., t$ define the set of points

$$S_u = \left\{ \left((10t)^u \cos \frac{2\pi j}{10t}, (10t)^u \sin \frac{2\pi j}{10t} \right) : j = 1, ..., u \right\}$$

and let $S = \bigcup_{u=1}^{t} S_u$. First of all, $|S_u| = u$ and the sets S_u are pairwise disjoint, so $|S| = t(t + 1)/2$. Clearly, the points of S_u lie on a circle of radius $(10t)^u$ and center 0.

As in the construction of Theorem 3.5, every triangle which has two vertices in some S_a and one vertex in some S_b, where $t \geq a > b \geq 1$, is acute. The proof is very similar so we omit it.

Suppose S has a partition $T_1, ..., T_k$ into obtuse sets, where $k < t$. For $i = 1, ..., k$ let a_i be the minimal integer such that $T_i \cap S_{a_i} \neq 0$. Then $|S_j \cap T_i| \leq 1$ for $j > a_i$, otherwise select $A \in S_{a_i} \cap T_i$ and $B, C \in S_j \cap T_i$, $B \neq C$, then ABC forms an acute triangle in T_i, which is impossible. As $k < t$, the set $\{1, ..., t\} \setminus \{a_1, ..., a_k\}$

is not empty. Let the maximum of this set be b. Then by the maximality of b there are at least $t-b$ elements in $\{a_1, \ldots, a_k\}$, which are larger than b, without the loss of generality we can suppose that $a_1, \ldots, a_{t-b} > b$. Then $T_1, \ldots, T_{t-b}$ and S_b are disjoint. Also, each of $T_{t-b+1}, \ldots, T_k$ can intersect S_b by at most a single element, so as $T_1, \ldots, T_k$ was a decomposition of S, we have $|S_b| \leq k-(t-b)$. However, this contradicts with $k < t$.

Hence, we showed that every partition of S into obtuse sets contains at least t obtuse sets. □

4. Random point sets

In this section, we shall give estimates on the size of the largest P_α-sequence in a random set. The construction of Theorem 3.2 for a point set with only small P_α-sets has a very special structure, it may be expected that a random set has much longer P_α-sequences. This is what we prove in the next theorem.

In what follows, we say that a sequence of events $(A_n)_{n=1}^{\infty}$ happens *with high probability* (*w.h.p.*), if

$$\lim_{n\to\infty} \mathbb{P}(A_n) = 1.$$

For simplicity, we shall not denote the index n, despite the fact that the event depends on n.

Before we proceed, we need the following version of Hoeffding's inequality [5]:

Theorem 4.7. (*Hoeffding's inequality*) *Let $\chi_1, \ldots, \chi_n$ be independent random variables such that $\chi_i \in [0,1]$. Let $\chi = \chi_1 + \ldots + \chi_n$. Then for any $t > 0$ we have*

$$\mathbb{P}(|\chi - E\chi| > t) < 2\exp\left(-\frac{2t^2}{n}\right).$$

Theorem 4.8. *Fix $0 < \alpha < \pi$. Choose N points randomly and independently inside the unit square, with uniform distribution. Let χ denote the length of the longest P_α-sequence formed by these points. There exists a constant $c = c(\alpha)$ such that $(\chi > c\sqrt{N})$ with high probability.*

Proof. Let the set of random points be H. Let $s = \lfloor\sqrt{N}\rfloor$ and divide the unit square into $s \times s$ little squares with sides having size $1/s$. Denote these squares by $S_{i,j}$ for $i,j = 1, \ldots, s$. Let M be the number of pairs (i,j) such that $|H \cap S_{i,j}| \geq 1$, so $M = \sum_{i,j=1}^{s} I(|S_{i,j} \cap H| \geq 1)$, where $I(\mathcal{A})$ is the indicator random variable of the event $\mathcal{A}$.

First, we show that $M > N/2$ holds with high probability. For every (i, j) and N large enough we have

$$\mathbb{P}(|S_{i,j} \cap H| \geq 1) = 1 - \left(1 - \frac{1}{s^2}\right)^N > 0.6,$$

thus $\mathbb{E}(M) > 0.6s^2$. Apply Hoeffding's inequality with $n = s^2, \chi_{s(j-1)+i} = I(|S_{i,j} \cap H| \geq 1)$ and $t = 0.05s^2$. Then

$$\mathbb{P}\left(M < \frac{N}{2}\right) < \mathbb{P}(|M - EM| > 0.05s^2) < 2e^{-2 \cdot 0.05^2 s^2},$$

so $M > N/2$ w.h.p.

By the pigeonhole principle, there exists an index $1 \leq k \leq s$ such that at least M/s squares in $S_{1,k}, ..., S_{s,k}$ contain an element of H. Let $a = \lceil 16/(\pi - \alpha) \rceil$ and for $l = 1, ..., a$ let

$$T_l = \bigcup_{j=0}^{\lfloor \frac{s-l}{a} \rfloor} S_{ja+l,k}.$$

Then $\bigcup_{l=1}^{a} T_l = \bigcup_{j=1}^{s} S_{j,,k}$, so by the pigeonhole principle again, there exist indices $1 \leq m \leq a$ and $j_1 < ... < j_r$ with $r \geq M/sa \geq M/\sqrt{N}a$ such that each of $S_{j_1 a+m,k}, \ldots$
$\ldots, S_{j_r a+m,k}$ contains an element of H. For $i = 1, \ldots, r$ let $X_i \in H \cap S_{j_i a+m,k}$.

We prove that $X_1, ..., X_r$ is a P_α-sequence, hence with the choice

$$c = \frac{1}{2\lceil 16/(\pi - \alpha) \rceil}$$

H contains a P_α-sequence with length at least $N/(2\sqrt{N}a) = c\sqrt{N}$ w.h.p.

For $i = 1, ..., r$ let $X_i = (x_i, y_i)$. Let $1 \leq u < v \leq r$ and let β be the absolute angle between $\overrightarrow{X_u X_v}$ and the vector $(1, 0)$. Then

$$\sin\beta = \frac{y_v - y_u}{\sqrt{(y_v - y_u)^2 + (x_v - x_u)^2}} = \frac{1}{\sqrt{1 + ((x_v - x_u)/(y_v - y_u))^2}}.$$

But $X_u \in S_{j_u a+m,k}$ and $X_v \in S_{j_v a+m,k}$, thus $|x_v - x_u| \geq (a-1)/s$ and $|y_v - y_u| \leq 1/s$. Hence

$$\sin\beta \leq \frac{1}{\sqrt{(a-1)^2+1}} < \frac{2}{a} \leq \frac{\pi-\alpha}{8}$$

and $\beta < (\pi-\alpha)/2$. So for any $1 \leq u < v < w \leq r$ the angle $X_u X_v X_w \angle$ is at least

$$\pi - 2 \cdot \frac{\pi - \alpha}{2} = \alpha,$$

hence $X_1, \ldots, X_r$ is a P_α-sequence. □

THEOREM 4.9. *Let $0 < \alpha < \pi$. Choose N points randomly and independently inside the unit square, with uniform distribution. Let χ denote the length of the longest P_α-sequence. There exists a constant $c = c(\alpha)$ such that*

$$(\chi < c\sqrt{N}\log N/\log\log N)$$

with high probability.

PROOF. Let the set of points be H, $s = \lceil\sqrt{N}\rceil$ and divide the unit square into $s \times s$ squares with size $1/s$, labelled as $S_{i,j}$, $i, j = 1, \ldots, s$. Let $b_{i,j} = |S_{i,j} \cap H|$, then for any pair (i, j) we have

$$\mathbb{P}(b_{i,j} > k) < \binom{N}{k}\frac{1}{s^{2k}} < \frac{1}{k!}.$$

Let $k = e\log N/\log\log N$, then

$$\mathbb{P}(b_{i,j} > k) < \frac{1}{k!} < \left(\frac{e}{k}\right)^k = \exp\left(-e\frac{\log N}{\log\log N}(\log\log N - \log\log\log N)\right) < \frac{1}{N^2}$$

if N is large enough. Say H is good, if $b_{i,j} < e\log N/\log\log N$ for all $i, j = 1, \ldots, s$. Then

$$\mathbb{P}(H\ good) > 1 - \sum_{i,j=1}^{s}\mathbb{P}\left(b_{i,j} > e\frac{\log N}{\log\log N}\right) > 1 - \frac{s^2}{N^2} > 1 - \frac{2}{N},$$

hence H is good w.h.p. In what follows we suppose that H is good.

Let $a = \lceil 16/\alpha \rceil$. For $k, l = 1, ..., a$ let

$$T_{k,l} = \bigcup_{i=0}^{\lfloor \frac{s-k}{a} \rfloor} \bigcup_{j=0}^{\lfloor \frac{s-l}{a} \rfloor} S_{ai+k,aj+l}$$

and define the map $\phi : T_{k,l} \to \mathbb{R}^2$ such that if $A \in S_{ai+k,aj+l}$, then $\phi(A) = (i, j)$ (ϕ is dependent on (k, l), but for simplicity we shall not denote it). First, we show that if $A, B \in T_{k,l}$ such that $\phi(A) \neq \phi(B)$, then the absolute angle between $\overrightarrow{AB}$ and $\overrightarrow{\phi(A)\phi(B)}$ is less than $\alpha/4$. Let β be the this angle. It is easy to see that β takes its maximum value if $\overrightarrow{AB} = ((a-1)/s, 1/s)$ or $(1/s, (a-1)/s)$ and $\overrightarrow{\phi(A)\phi(B)} = (1, 0)$ or $(0, 1)$ respectively. In these cases

$$\sin \beta = \frac{1}{\sqrt{(a-1)^2 + 1}} < \frac{2}{a} \leq \frac{\alpha}{8},$$

so $\beta < \alpha/4$. This means that if $A, B, C \in T_{k,l}$ such that $\phi(A), \phi(B), \phi(C)$ is pairwise different, then

$$|ABC\angle - \phi(A)\phi(B)\phi(C)\angle| < \frac{\alpha}{2}.$$

Hence, if $Y_1, ..., Y_r$ is a P_α-sequence in $T_{k,l}$ such that $\phi(Y_1), ..., \phi(Y_r)$ is pairwise different, then $\phi(Y_1), ..., \phi(Y_r)$ is a $P_{\alpha/2}$-sequence. But $\phi(Y_1), ..., \phi(Y_r)$ are elements of a $(\lfloor (s-k)/a \rfloor + 1) \times (\lfloor (s-l)/a \rfloor + 1)$ square grid, so by Theorem 3.3 we have $r < c_{\alpha/2} \lceil s/a \rceil$ with some constant $c_{\alpha/2}$. So $r < \alpha c_{\alpha/2} \sqrt{N}/8$.

Let $Z_1, ..., Z_m$ be a P_α-sequence in $T_{k,l} \cap H$, $S = \{Z_1, ..., Z_m\}$. Then $|\phi(S)| < < \alpha c_{\alpha/2} \sqrt{N}/8$ and as H is good, every element in $\phi(S)$ has an inverse image of size at most $e \log N / \log \log N$, hence

$$m < \frac{\alpha}{8} c_{\alpha/2} e \sqrt{N} \frac{\log N}{\log \log N}.$$

Finally, let $X_1, ..., X_n$ be a P_α-sequence in H. Then for all $k, l = 1, ..., a$ the intersection of $T_{k,l}$ and the sequence contains at most

$$\frac{\alpha}{8} c_{\alpha/2} e \sqrt{N} \frac{\log N}{\log \log N}$$

points, so

$$n < a^2 \frac{\alpha}{8} c_{\alpha/2} e\sqrt{N} \frac{\log N}{\log \log N} < \frac{1000}{\alpha} c_{\alpha/2} \sqrt{N} \frac{\log N}{\log \log N}.$$

Set $c = 1000\alpha c_{\alpha/2}/\alpha$, then $\chi < c\sqrt{N} \log N / \log \log N$ with high probability. □

5. P_α-sets and P_α-sequences in higher dimensions

In higher dimensions things are getting complicated. Many of our previous results can be generalized for dimension $d \geq 3$, however there are huge gaps between the lower and upper bounds produced. This is due to the sphere-packing and sphere-covering problem. We are going to state one of the weak estimates.

Proposition 5.1. *Let $\alpha > 0$ real.*

(i) *In the d-dimensional Euclidean space, there exist vectors $v_1, \ldots, v_k$ with $k \leq (4/\alpha)^{d-1}$ such that for any vector w one can choose $1 \leq i \leq k$ such that the absolute angle between v_i and w is less than $\alpha/2$.*

(ii) *In the d-dimensional Euclidean space, there exist lines $l_1, \ldots, l_m$ with $m \geq (1/\alpha)^{d-1}$ such that for any $1 \leq i < j \leq m$ the angle between l_i and l_j is more than α.*

There are better estimations than the ones stated in Proposition 5.1, but these results can be easily verified by the reader himself. These bounds were used by Erdős and Füredi [3] to prove the following theorem: let $\alpha_d(n)$ denote the infimum of the largest angle formed by three points in an n element set in the d-dimensional space. Then

$$\pi \left(1 - \frac{1}{\sqrt[d-1]{\log_2 n}}\right) < \alpha_d(n) < \pi \left(1 - \frac{4}{\sqrt[d-1]{\log_2 n}}\right).$$

We show how their method can be generalized to find a large P_α-sequence and to construct sets with only small P_α-sets.

Definition 5.1. *Let $D \subset \mathbb{R}^d$ be an open convex cone (if $v \in D$ and $\lambda > 0$ then $\lambda v \in D$, if $v, w \in D$ then $v + w \in D$) such that D and $-D = \{-v : v \in D\}$ are disjoint. Define the relation $<_D$ in the following way: for $X, Y \in \mathbb{R}^d$ let $X <_D Y$ if $\overrightarrow{XY} \in D$. Furthermore, denote by $\alpha(D)$ the supremum of the absolute angles between all pairs of vectors in D.*

It is easy to see that $<_D$ is a partial order on $\mathbb{R}^d$. Furthermore, a $<_D$-chain is a $P_{\pi-\alpha(D)}$-sequence.

THEOREM 5.10. *Let $0 < \alpha < \pi$ real, d, t integers.*

(i) *Let $k = \lceil (4/(\pi-\alpha))^{d-1} \rceil$. Then any set $H \subset \mathbb{R}^d$, $|H| \geq t^k + 1$ contains a P_α-sequence with at least $t+1$ elements.*

(ii) *Let $m = \lfloor (1/(\pi-\alpha))^{d-1} \rfloor$. Then there exists a set $H_0 \subset \mathbb{R}^d$, $|H_0| = t^m$ such that any P_α set of H_0 has at most t elements.*

PROOF. (i) According to Proposition 5.1 (i) we can select vectors $v_1, \ldots, v_k \in \mathbb{R}^d$ such that for any w there exist $1 \leq i \leq k$ such that the angle between v_i and w is less than $(\pi - \alpha)/2$. For $i = 1, \ldots, k$ define D_i as the set of nonzero vectors w such that the absolute angle between w and v_i is smaller than $(\pi-\alpha)/2$. Then $\alpha(D_i) = \pi - \alpha$. Moreover $(H, <_{D_1}, \ldots, <_{D_k})$ is a multi-poset, so by Lemma 2.1 H contains a $<_{D_j}$-chain with at least $t+1$ elements for some $1 \leq j \leq k$, which is a P_α-sequence.

(ii) According to Proposition 5.1 (ii) we can select lines $l_1, \ldots, l_m$ such that for any $1 \leq i < j \leq m$ the smaller angle between l_i and l_j is less than $\pi - \alpha$. Choose unit vectors $e_1, \ldots, e_m$ such that e_i is parallel to l_i. Let $l > 0$ be real. Define the set

$$H_0 = \left\{ \sum_{j=1}^{m} a_j l^j e_i : a_1, \ldots, a_m \in [t] \right\}.$$

Applying the same proof as in Theorem 3.2, if l is large enough H_0 does not contain a P_α-set with more than t elements. □

Again, we strengthen our results for obtuse sets and sequences.

THEOREM 5.11.

If $H \subset \mathbb{R}^d$, $|H| \geq t^{2^{d-1}} + 1$, then H contains an obtuse sequence with at least $t+1$ elements.

There exists a set $H_0 \subset \mathbb{R}^d$, $|H_0| = \binom{t+d-1}{d}$ such that any obtuse set of H_0 has at most t elements.

PROOF. (i) Take an orthogonal coordinate system such that none of the vectors $\overrightarrow{XY}$, $\overrightarrow{0X}$, $\overrightarrow{0Y}$ has a zero coordinate, where $X, Y \in H$. For any $\mathbf{e} = (e_1, \ldots, e_d) \in \{-1, 1\}^d$

define the convex cone

$$D_{\mathbf{e}} = \{(v_1, ..., v_d) : v_i e_i > 0, i = 1, ..., d\}.$$

Then $\alpha(D_{\mathbf{e}}) = \pi/2$, because the usual scalar product of any two of its vectors is nonnegative. Let $<_1, ..., <_{2^{d-1}}$ be all the different relations $<_{D_{\mathbf{e}}}$ for all $\mathbf{e} \in \{-1, 1\}^d$, $e_1 = 1$. Then $(H, <_1, ..., <_{2^{d-1}})$ is a multi-poset and a $<_i$-chain is an obtuse sequence. Applying Lemma 2.1 proves the statement.

(ii) We prove the following: there exists $H_0 \subset \mathbb{R}^d$, $|H_0| = \binom{t+d-1}{d}$ such that any obtuse set of H_0 has at most t elements and none of the angles determined by three points of H_0 is $\pi/2$. We prove this by induction on d. For $d = 2$ the construction of Theorem 3.6 suffices. Now suppose $d \geq 3$.

Take an Euclidean coordinate system in $\mathbb{R}^d$. For $u = 1, ..., t$ let $H_u \subset \mathbb{R}^{d-1}$, $|H_u| = \binom{u+d-2}{d-1}$ with largest obtuse set having at most u elements and none of the angles is $\pi/2$. Embed H_u isometrically into the hyperplane $S = \{x_d = 0\}$, let the image be H'_u. For $r > 0$ let B_r denote the sphere with radius r and center 0. Define the map $\phi_r : S \to \mathbb{R}^d$ as

$$\phi_r((x_1, ..., x_{d-1}, 0)) = (x_1, ..., x_{d-1}, \sqrt{r^2 - x_1^2 - ... - x_{d-1}^2})$$

if $(x_1, ..., x_{d-1}, 0)$ is inside B_r, and $\phi_r((x_1, ..., x_{d-1}, 0)) = (x_1, ..., x_{d-1}, 0)$ otherwise. Then ϕ_r maps the points of S inside B_r into B_r.

For every $\epsilon > 0$ there exists $R_u(\epsilon)$ such that for $r > R_u(\epsilon)$ the inequalities

$$1 \leq \frac{|\phi_r(X)\phi_r(Y)|}{|XY|} < 1 + \epsilon$$

and $|XYZ\angle - \phi_r(X)\phi_r(Y)\phi_r(Z)\angle| < \epsilon$ hold for every pairwise different $X, Y, Z \in H'_u$. Choose r_u satisfying the following lower bounds:

(1) the sphere with center 0 and radius r_u contains H'_u;

(2) $r_u > R_u(1)$ and $r_u > 20 \max_{X,Y \in H'_u} |XY|$;

(3) there exists $\eta > 0$ such that none of the angles $XYZ\angle$ $(X, Y, Z \in H'_u)$ fall in $(\pi/2 - \eta, \pi/2 + \eta)$; let $r_u > R_u(\eta)$.

Let $H^*_u = \phi_{r_u}(H'_u)$ and let $A_1 = H^*_1$. For $i \geq 2$ we define A_i as a scaled image of H^*_i. We calculate the ratio of the scaling in the following way: suppose A_{i-1} lies on a sphere with radius R_{i-1} and center 0. Let the smallest distance occurring between two different points of H_i^* be s. Scale H_i^* from the origin with ratio

$$R_{i-1} \max\left\{\frac{2}{s}, \frac{10}{r_u}\right\}$$

and let the image be A_i. Note that the sets $A_1, ..., A_t$ lie on concentric spheres with radius $R_1 < ... < R_t$, and by the choice of the scaling ratio the smallest distance between any two different points of A_i is at least $2R_{i-1}$.

Define $H_0 = \bigcup_{i=1}^{t} A_i$. Then,

$$|H_0| = \sum_{i=1}^{t} \binom{i+d-2}{d-1} = \binom{t+d-1}{d}.$$

Also, we show that if $X, Y \in A_a$, $X \neq Y$ and $Z \in A_b$ with $b < a$, then the triangle XYZ is acute. The set of points T for which $XYT\angle < \pi/2$ and $TXY\angle < \pi/2$ is an open strip bounded by two parallel hyperplanes going through X and Y and perpendicular to $\overrightarrow{XY}$. As $|XY| \geq 2R_{a-1} > R_b$ and X, Y lie on a sphere with center 0, this strip contains the sphere with radius R_b and center 0, so it contains $Z \in A_b$. It only remains to show that $XZY\angle < \pi/2$. Otherwise the largest side of the triangle would be XY, but

$$|XZ| \geq R_a - R_b \geq R_a - R_{a-1} \geq \frac{9}{10}R_a$$

and $|XY| \leq R_a/10$ by lower bound (2) on r_a. So $XZY\angle < \pi/2$.

Finally, let $S \subset H_0$ be an obtuse set. Let m be the smallest integer such that $A_m \cap S \neq 0$. Then $|S \cap A_l| \leq 1$ for all $l > m$ by the previous thoughts and $|S \cap A_m| \leq m$ as $|A_m| = m$. So $|S| \leq m + (t - m) = t$, which proves (ii). □

The result of (ii) is not sharp. In the previous construction, if we replace $A_1 \cup A_2$ with a proper scaling of the cube $\{0, 1\}^d$, the largest obtuse set also has at most t elements and $|H_0| = \binom{t+d-1}{d} - (d+1) + 2^d$. Furthermore, if we start the induction at $d = 2$ with the construction of Theorem 3.5 instead of Theorem 3.6, we also get a stronger result. However, our conjecture is that the exponent of t is right, so any set with more than ct^d points contains an obtuse set with $t + 1$ elements, where c is some constant.

Finally, we mention a corresponding bound for random sets.

THEOREM 5.12. *Let* $0 < \alpha < \pi$ *be real,* $d \geq 2$ *integer. Choose* N *points inside the* d*-dimensional unit cube randomly and independently, with uniform distribution. Let* χ *denote the length of the longest* P_α*-sequence formed by these points. Then there exists a constant* $c = c(d, \alpha)$ *such that* $(\chi > c\sqrt[d]{N})$ *with high probability.*

PROOF. Almost identical to the proof of Theorem 4.8. □

6. Open questions

For $0 < \alpha < \pi$ let $A_\alpha(N)$ denote the maximal n such that every set in the plane with N points contains a P_α-set with n points.

Similarly, let $B_\alpha(N)$ be the maximal n such that every set in the plane with N points contains a P_α-sequence with length n.

CONJECTURE 6.1. $A_{\pi/2}(N) = B_{\pi/2}(N) = (\sqrt{2} + o(1))\sqrt{N}$.

CONJECTURE 6.2. There exists a positive constant c such that the following holds. For every N there exists an obtuse set $H \subset \mathbb{R}^2$ such that $|H| = N$ and H has no obtuse sequence with length more than $c\sqrt{N}$.

OPEN QUESTION 1. *In Theorem* 3.1 *and Theorem* 3.2 *we prove that if* $\pi - \pi/n < \alpha \leq \pi - \pi/(n+1)$, *then*

$$N^{\frac{1}{n+1}} \leq A_\alpha(N) \leq B_\alpha(N) \leq N^{\frac{1}{n}}.$$

What is the right exponent if $\alpha > \pi/2$*? Namely, what is*

$$\lim_{N\to\infty} \frac{\log A_\alpha(N)}{\log N}$$

and

$$\lim_{N\to\infty} \frac{\log B_\alpha(N)}{\log N}?$$

OPEN QUESTION 2. *Let*

$$C_\alpha = \limsup_{N\to\infty} \frac{B_\alpha(N)}{\sqrt{N}}.$$

According to Theorem 3.3, for $0 < \alpha < \pi/2$ we have $C_\alpha < \infty$. Is it true that $C_\alpha \to \infty$ as $\alpha \to 0$? What is the exact value of C_α?

Conjecture 6.3. Let t be a positive integer. Then every set $H \subset \mathbb{R}^d$, $|H| = t^d + 1$ contains an obtuse sequence with $t+1$ elements (for $t = 2$ it is the theorem of Danzer and Grünbaum).

Conjecture 6.4. Let t be a positive integer. For every $\pi > \alpha > 0$ there exists a constant $c = c(d, \alpha) > 0$ such that the longest P_α-sequence in the cube grid $[t]^d \subset \mathbb{R}^d$ has at most ct elements.

Conjecture 6.5. Let $d \geq 2$ be an integer and $0 < \alpha < \pi$. Choose N points randomly and independently inside the d-dimensional unit cube, with uniform distribution. Let χ denote the size of the largest P_α-sequence formed by these points. There exists a constant $c = c(\alpha, d)$ such that for every $\epsilon > 0$ the event

$$((c-\epsilon)\sqrt[d]{N} < \chi < (c+\epsilon)\sqrt[d]{N})$$

holds with high probability.

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